Complex Analysis

In the last class, we did the following theorem:

Theorem (Page 142 sec 44) Suppose that the function $f(z)$ is continuous on a domain D. Then following statements are equivalent

(a) $f(z)$ has an antiderivative $F(z)$ throughout D.

(b) the integrals of $f(z)$ along contours lying entirely in D and extending from fixed point z_1 to any fixed point z_2 all have the same value, namely

$$\int_{z_1}^{z_2} f(z)\,dz = F(z)\Big]_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F(z)$ is the antiderivative ~~of f(z)~~ in (a).

(c) the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.

Remark 1. It is important to note that theorem doesn't claim that any of these statements is true for a given function $f(z)$.

It says that all of them are true or none of them is true.

Remark 2 Examples 1, 2 and 3 were also discussed in the class.

Problems based on sec 44 and sec 45 (Page 149)

Q1. Use an antiderivative to show that for every contour C extending from a point z_1 to a point z_2

$$\int_C z^n = \frac{1}{n+1}\left(z_2^{n+1} - z_1^{n+1}\right), \quad (n = 0, 1, 2, 3, \ldots).$$

Soln. Let $f(z) = z^n$

& $F(z) = \dfrac{z^{n+1}}{n+1}$.

Then $F'(z) = \dfrac{d}{dz}\left(\dfrac{z^{n+1}}{n+1}\right) = (n+1)\dfrac{z^n}{(n+1)} = z^n = f(z)$.

By Antiderivative Theorem,

$$\int_C f(z)\,dz = F(z)\Big]_{z_1}^{z_2} = \frac{\left(z_2^{n+1} - z_1^{n+1}\right)}{n+1}, \quad (n = 0, 1, 2, \ldots)$$

Q2. Evaluate (a) $\displaystyle\int_i^{i/2} e^{\pi z}\,dz$ (b) $\displaystyle\int_0^{\pi+2i} \cos(z/2)\,dz$

(c) $\displaystyle\int_1^3 (z-2)^3\,dz$.

(a) Let $f(z) = e^{\pi z}$ & $F(z) = \dfrac{e^{\pi z}}{\pi}$

Since $\dfrac{e^{\pi z}}{\pi}$ is an entire function

and $\dfrac{d}{dz}\left(\dfrac{e^{\pi z}}{\pi}\right) = e^{\pi z} = f(z)$

$\therefore F(z) = \dfrac{e^{\pi z}}{\pi}$ is an antiderivative of $f(z) = e^{\pi z}$.

By Antiderivative Theorem,

$$\int_i^{i/2} e^{\pi z}\,dz = \frac{e^{\pi z}}{\pi}\Big]_i^{i/2} = \frac{1}{\pi}\left[e^{\pi i/2} - e^{\pi i}\right]$$

$$= \frac{1}{\pi}\left[i - (-1)\right] = \frac{i+1}{\pi}.$$

(b) $\displaystyle\int_0^{\pi+2i} \cos(z/2)\,dz$

Let $f(z) = \cos z/2$ & $F(z) = 2\sin(z/2)$.

Since $2\sin(z/2)$ is an entire function

and $\frac{d}{dz}\left(2\sin z/2\right) = \cos z/2$

$$\int_0^{\pi+2i} f(z)dz = F(\pi+2i) - F(0) = 2\left[\sin\left(\frac{\pi+2i}{2}\right) - \sin 0\right]$$

$$= 2\left[\sin(\pi/2 + i) - \sin 0\right]$$

$$= 2\cos i = 2\times \frac{(e^{i^2}+e^{-i^2})}{2}$$

$$= e^{-1} + e$$

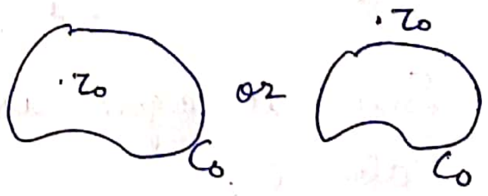
$$= 1/e + e.$$

$$\boxed{\text{Note that, } \cos z = \frac{e^{iz}+e^{-iz}}{2},\ \sin z = \frac{e^{iz}-e^{-iz}}{2i}}$$

(c) Do yourself.

Q3. Show that $\displaystyle\int_{C_0} (z-z_0)^{n-1} dz = 0$ $(n = \pm 1, \pm 2, --)$ when C_0 is any closed contour which doesn't pass through the point z_0.

Soln. ~~Note that~~



Since $z_0 \notin C_0$

$\therefore$ $z - z_0 \neq 0$ $\forall z \in C_0$

$\therefore$ $F(z) = \dfrac{(z-z_0)^n}{n}$ is analytic $\forall z \in C_0$

$n = \pm 1, \pm 2, \pm 3, --$

Moreover; $\dfrac{d}{dz}(F(z)) = \dfrac{d}{dz}\left(\dfrac{(z-z_0)^n}{n}\right) = \dfrac{n.(z-z_0)^{n-1}}{n} = (z-z_0)^{n-1}$

$\therefore \quad \frac{d}{dz}(F(z)) = f(z), \quad \forall z \in C_0$

Since C_0 is closed contour

$\therefore$ By Antiderivative Theorem.

$$\int_{C_0} f(z)\,dz = 0.$$

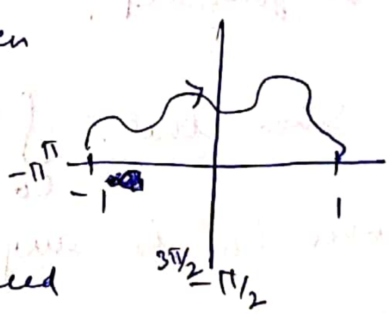
Q5. Show that $\displaystyle\int_{-1}^{1} z^i\,dz = \frac{1+e^{-\pi}}{2}(1-i)$

where the integrand denotes the principal branch

$$z^i = \exp(i \operatorname{Log} z) \qquad (|z|>0,\ -\pi < \operatorname{Arg} z < \pi)$$

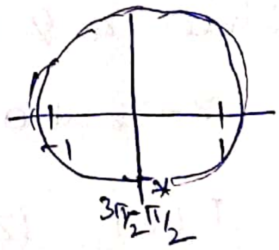
of z^i. and where the path of integration is any contour from $z=-1$ to $z=1$, that, except for its end points, lies above the real axis.

Soln: An antiderivative of the given branch cannot be used since the branch is not defined at $z=-1$ but the integrand can be replaced by the branch



$$z^i = \exp(i \log z) \qquad (|z|>0,\ -\pi/2 < \arg z < 3\pi/2)$$

Since it agrees with the integrand along C.



Use an antiderivative $F(z) = \dfrac{z^{i+1}}{i+1}$ of this new branch.

$$I = \int_{-1}^{1} z^i\,dz = \left[\frac{z^{i+1}}{i+1}\right]_{-1}^{1} = \frac{1}{i+1}\left[(1)^{i+1} - (-1)^{i+1}\right]$$

$$= \frac{1}{i+1}\left[e^{(i+1)\log 1} - e^{(i+1)\log(-1)}\right]$$

$$= \frac{1}{i+1}\left[e^{(i+1)(\ln|1| + i\cdot 0)} - e^{(i+1)(\ln|-1| + i\pi)}\right]$$

$$(\because \log z = \ln|z| + i\theta)$$

$$= \frac{1}{i+1}\left[e^{(i+1)(0)} - e^{(i+1)(i\pi)}\right]$$

$$= \frac{1}{i+1}\left[e^{0} - e^{-\pi}\cdot e^{i\pi}\right]$$

$$= \frac{1}{i+1}\left[1 + e^{-\pi}\right] \quad (\because e^{i\pi} = \cos\pi + i\sin\pi = -1)$$

$$= \frac{(1-i)}{(1-i)(1+i)}\left[1+e^{-\pi}\right]$$

$$= \frac{1}{2}\left(1+e^{-\pi}\right)(1-i).$$

Sec 46

Cauchy - Goursat Theorem

If a function 'f' is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0$$

Remark: Proof is not in the syllabus.

Problems on Page 160

Remark: Only Ques 1 is in the syllabus.

Q 1. Apply the Cauchy-Goursat theorem to show that

$$\int_C f(z)\,dz = 0$$

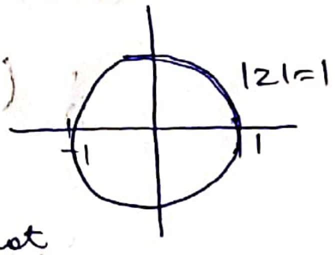
when the contour C is the unit circle $|z|=1$, in either direction and when

(a) $f(z) = \dfrac{z^2}{z-3}$ (b) $f(z) = z e^{-z}$ (c) $f(z) = \dfrac{1}{z^2+2z+2}$

(d) $f(z) = \operatorname{sech} z$ (e) $f(z) = \tan z$ (f) $f(z) = \operatorname{Log}(z+2)$.

Soln. Note that contour C is

(a) $f(z) = \dfrac{z^2}{z-3}$



The only point where f is not analytic is $z=3$.

But $z=3$ neither lies inside nor on the contour $|z|=1$.

$\therefore$ f is analytic at all the points interior to and on $|z|=1$ which is a simple closed contour

$\therefore$ By Cauchy-Goursat Theorem,

$$\int_C \frac{z^2}{z-3}\,dz = 0.$$

(b) $f(z) = z e^{-z} = \frac{z}{e^z}$

$f(z)$ is entire function.

$\therefore$ f is analytic ~~inside~~ at all the points inside to and on $|z|=1$.

Moreover $|z|=1$ is simple closed contour

$\therefore$ By Cauchy - Goursat Theorem

$$\int_C z e^{-z} dz = 0.$$

(c) $f(z) = \frac{1}{z^2+2z+2}$

The only points where f is not analytic can be obtained by putting ~~zero~~ $z^2+2z+2=0$,

$$\Rightarrow z = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\Rightarrow |z| = \sqrt{2} > 1$$

$\therefore$ $z = -1+i, -1-i$ lie outside $|z|=1$

$\therefore$ f is analytic at all the points inside to and on $|z|=1$

Moreover, $|z|=1$ is a simple closed contour

$\therefore$ By Cauchy - Goursat Theorem

$$\int_C \frac{1}{z^2+2z+2} dz = 0.$$

(d) $f(z) = \operatorname{sech} z$

$$f(z) = \operatorname{sech} z = \frac{1}{\cosh z}$$

The only points where f is not analytic can be obtained by putting $\cosh z = 0$

$$\Rightarrow \quad z = \left(\tfrac{\pi}{2} + n\pi\right) i \quad (n = 0, \pm 1, \pm 2, \ldots)$$

(see eqn. 15 on Page 111).

For any $n = 0, \pm 1, \pm 2, \ldots$

$$|z| = \left|\left(\tfrac{\pi}{2} + n\pi\right) i\right|$$
$$= \left|\tfrac{\pi}{2} + n\pi\right|$$

It can be easily observed that

$|z| > 1$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

(Verify it).

$\therefore$ f is analytic at all the points interior to and on $|z| = 1$.

Moreover $|z| = 1$ is a simple closed contour

$\therefore$ By Cauchy–Goursat Theorem

$$\int_C \operatorname{sech} z \, dz = 0.$$

(e) $f(z) = \tanh z$ (Do yourself)

Refer eqn. 16 on Page 111 and part (d) above)

(f) $f(z) = \log(z+2)$

Hint: $\boxed{f'(z) = \frac{1}{z+2}.}$

$$\boxed{\begin{aligned} f(z) &= \operatorname{Log} z = \ln|z| + i\theta = \underbrace{\ln\sqrt{x^2+y^2}}_{u(x,y)} + i\underbrace{\tan^{-1}(y/x)}_{v(x,y)} \\ \text{Then } f'(z) &= u_x + i v_x = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} = \frac{x-iy}{x^2+y^2} \end{aligned}} = \frac{1}{x+iy} = \frac{1}{z}$$